Lecture 4 Summary

PHYS798S Spring 2016

February 11, 2016

The Macroscopic Quatum Model of Superconductivity

We do this before developing the MICRO-scopic theory of superconductivity.

The key statement is the following: Superconductivity is inherently a quantum mechanical phenomenon that manifests itself on macroscopic scales.

So develop a macroscopic quantum model to "explain" superconducting phenomena.

0.1 Review of QM

Review of Basic Quantum Mechanics for single particles:
Time-dependent Schrodinger equation
Probability amplitude for finding the particle
Normalization condition on the wavefunction
Probability current
Continuity equation for probability density
Charged particle under the influence of electric anf magnetic fields, with associated scalar and vector potentials
Schrodinger equation including ϕ and $\overrightarrow{A}$
Probability current including $\overrightarrow{A}$

0.2 Macroscopic Quantum Treatment of Superconductors

Hypothesis: There exists a macroscopic quantum wavefunction $\Psi(\overrightarrow{r}, t)$ that describes the behavior of the entire ensemble of super-electrons in the superconductor.
Here $\Psi(\overrightarrow{r}, t)$ is a field-like quantity that describes the coherent behavior of the super-electrons.
Normalization constraint for the Macroscopic Quantum Wave Function (MQWF): $\int \Psi^*(\overrightarrow{r}, t)\Psi(\overrightarrow{r}, t)dV = N^*$, where N^* is the total number of super-electrons that the MQWF describes. Note that * is NOT complex conjugation here (N is real)!
Therefore, the local density of super-electrons is $\Psi^*(\overrightarrow{r}, t)\Psi(\overrightarrow{r}, t) = n^*(\overrightarrow{r}, t)$. Note that $|\Psi(\overrightarrow{r}, t)|^2$ is no longer a probability but in fact describes the location of a sub-set of all of the super-electrons.

Thus the flow of probability $\overrightarrow{J}_{prob}$ now describes an actual flow of particles, or a true physical current. We can write the super-current density as $\overrightarrow{J}_s = q^*\Re\left\{\Psi^*\left(\frac{\hbar}{\imath m^*}\overrightarrow{\nabla} - \frac{q^*}{m^*}\overrightarrow{A}\right)\Psi\right\}$. We take the super-electrons to have charge q^*, mass m^*, and density n^*, all real quantities.

In polar format, we expect the MQWF to be of the form $\Psi(\overrightarrow{r},t) = \sqrt{n^*(\overrightarrow{r},t)}e^{\imath\theta(\overrightarrow{r},t)}$, where $n^* = \Psi^*\Psi$ and $\theta(\overrightarrow{r},t)$ is a real phase factor. Putting this version of Ψ in to the current density expression, we find $\overrightarrow{J}_s = q^*n^*(\overrightarrow{r},t)\left(\frac{\hbar}{m^*}\overrightarrow{\nabla}\theta(\overrightarrow{r},t) - \frac{q^*}{m^*}\overrightarrow{A}(\overrightarrow{r},t)\right)$.

Or, using $\overrightarrow{J}_s(\overrightarrow{r},t) = n^*(\overrightarrow{r},t)q^*\overrightarrow{v}_s$, we can write for the super-fluid velocity $\overrightarrow{v}_s = \frac{\hbar}{m^*}\overrightarrow{\nabla}\theta(\overrightarrow{r},t) - \frac{q^*}{m^*}\overrightarrow{A}(\overrightarrow{r},t)$. Hence the (measurable) superfluid current density is related to the gradient of the phase of the MQWF and the vector potential, neither of which can be directly measured!

The vector potential reproduces the (measurable) magnetic field $\overrightarrow{B}$ through its curl $\overrightarrow{B} = \overrightarrow{\nabla}\times\overrightarrow{A}$, but it can be modified by the gradient of any real scalar function of position and produce the same magnetic field: $\overrightarrow{A}\rightarrow\overrightarrow{A}' = \overrightarrow{A} + \overrightarrow{\nabla}\chi$. This flexibility in gauge choice also constrains the MQWF phase through $\theta\rightarrow\theta' = \theta + \frac{q^*}{\hbar}\chi$. With this change of gauge one can show that the supercurrent density $\overrightarrow{J}_s(\overrightarrow{r},t)$ is gauge invariant.

0.3 Generalized London relation

Taking $m^* = 2m$, $q^* = -2e$ and $n^* = n/2$ one can see that $\Lambda^* = \Lambda$! This allows us to write the generalized London relation as follows
$\Lambda\overrightarrow{J}_s = \frac{\hbar}{q^*}\overrightarrow{\nabla}\theta - \overrightarrow{A}$.
Taking the curl of both sides gives the second London equation. Note that the "quantum mechanics" drops out when the curl is taken!
Taking the time derivative of both sides of the London relation gives the first London equation once the phase of the MQWF is interpreted as an energy and the gradient gives the electric field derived from the electric potential ϕ.

0.4 Fluxoid Quantization

Consider a closed contour C that is entirely within a superconductor. Integrate the generalized London relation around this contour:
$\oint_C\left(\Lambda\overrightarrow{J}_s\right)\cdot d\overrightarrow{l} = \frac{\hbar}{q^*}\oint_C\overrightarrow{\nabla}\theta\cdot d\overrightarrow{l} - \oint_C\overrightarrow{A}\cdot d\overrightarrow{l}$
We can use Stoke's theorem on the last term (only). This last term yields the magnetic flux through any surface S that terminates on the contour C: $\oint_C\overrightarrow{A}\cdot d\overrightarrow{l} = \iint_S\overrightarrow{\nabla}\times\overrightarrow{A}\cdot d\overrightarrow{S} = \iint_S\overrightarrow{B}\cdot d\overrightarrow{S} = \Phi_S$.
The middle term is the integral of the gradient of the phase of the MQWF. With careful analysis noting the 2π ambiguity of the phase, one finds that the integral becomes: $\frac{\hbar}{q^*}\oint_C\overrightarrow{\nabla}\theta\cdot d\overrightarrow{l} = \frac{\hbar}{q^*}2\pi p$, where p can be any positive or negative integer, or zero.

Now we have: $\oint_C \left(\Lambda \overrightarrow{J}_s\right) \cdot d\overrightarrow{l} + \iint_S \overrightarrow{B} \cdot d\overrightarrow{S} = \frac{h}{q^*}p$. This is a statement of "fluxoid quantization". The left hand side of the equation is the fluxoid, and the right hand side is a special combination of fundamental constants known as the flux quantum, $\Phi_0 = h/2e$ where h is Planck's constant and e is the electronic charge. The factor of 2 was put in by hand here, but it is the value seen in experiments on trapped flux in superconductors.

Note that only in the case where the contour C is chosen in such a way that the current contour integral is zero do you have the special case of "flux quantization." One way to do this is to have a multiply connected superconductor in whch C is chosen deep inside the superconductor such that $J_s = 0$ there. Then the flux through any surface S that terminates on C will be quantized in units of Φ_0.

The class web site shows data for the trapped flux in a superconducting cylinder as a function of applied magnetic field. The discrete steps in magnetic moment of the trapped flux is a clear sign of flux quantization.